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Strain gradient theory in orthogonal curvilinear coordinates

Jidong Zhao *, Dorival Pedroso

Centre for Geotechnical and Materials Modelling, University of Newcastle, University Drive, Callaghan, NSW 2308, Australia

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Abstract

In this short note, general formulations of the Toupin–Mindlin strain gradient theory in orthogonal curvilinear coordinate systems are derived, and are then specified for the cases of cylindrical coordinates and spherical coordinates. Expressions convenient for practical use are presented for the corresponding equilibrium equations, boundary conditions, and the physical components for strains and strain gradients in the two coordinate systems. The results obtained in this paper are general and complete, and can be useful for a wide range of applications, such as asymptotic crack tip field analysis, cylindrical and spherical cavity expansion problems, and the interpretation of micro/nano indentation tests and bending/twisting tests on small scales.

Keywords: Strain gradient theory; Orthogonal curvilinear coordinates; Cylindrical coordinates; Spherical coordinates

1. Introduction

In early 1960s, Toupin (1962) and Mindlin (1964) proposed a strain gradient theory in which the strain energy function is assumed to depend on both the strain and strain gradient. Numerous early extensions of this theory have since been developed and used for various applications (see, e.g., Toupin, 1964; Mindlin, 1965; Mindlin and Eshel, 1968; Bleustein, 1966; Bleustein, 1967; Eringen, 1968; Eshel and Rosenfeld, 1970, 1975; Germain, 1973). The past two decades witness a revived interest in this theory from the broad community of mechanics and material science. Based on this theory, a variety of new models have been developed to investigate such problems as strain localisation and size effects in materials and challenging issues on the micro/nano scales. Conventional continuum theories fail to handle these problems due to the lack of intrinsic length scales that represent the measures of microstructure in their constitutive relations (see, e.g., Fleck and Hutchinson, 1993, 1997, 2001; Chambon et al., 1996, 1998, 2001, 2004; Georgiadis et al., 2000; Georgiadis and Grentzelou, 2006; Zhao et al., 2005, 2006, 2007a,b; Zhao and Sheng, 2006, and references cited therein). Therefore, being one of the most complete linear generalised continuum theories as commented by Georgiadis et al. (2000), the Toupin–Mindlin strain gradient theory has evidently enjoyed great success so far, and will be

* Corresponding author. Tel.: +61 2 4921 5741; fax: +61 2 4921 6991.
E-mail address: Jidong.zhao@newcastle.edu.au (J. Zhao).

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chosen for the study in this paper. Meanwhile, it is noteworthy that there are many other gradient theories that have also received much attention from many engineering fields. Amongst them, the gradient plasticity theory pioneered by Aifantis and co-workers (Aifantis, 1984; Zbib and Aifantis, 1988; Vardoulakis and Aifantis, 1991) is one of the most widely used. The Aifantis theory considers the Laplacian of plastic strain or other internal variables in the consistency conditions and/or flow rule, which marks its key difference with the Toupin–Mindlin theory. In this note, however, we do not attempt to make a comprehensive comparison among the various gradient theories, for which purpose the readers are referred to more recent papers such as that by Chambon et al. (2004).

The original Toupin–Mindlin Strain Gradient Theory (abbreviated hereafter as SGT) and most models based on it have been formulated in general tensor forms, which, in theory, can be recast to any specific formulations if necessary. In dealing with applications where rectangular cartesian coordinates are appropriate, one may find it straightforward and trivial to obtain the specific formulations in terms of rectangular coordinates. However, when strain gradient theories are to be used in cases where curvilinear coordinates are suitable, the corresponding formulations regarding the equilibrium equations and boundary conditions can not be obtained automatically, and the course of derivation is always exceedingly complicated yet tedious and painful. Meanwhile, formulations of strain gradient theories under orthogonal curvilinear coordinates such as cylindrical or spherical coordinates are particularly useful for a wide range of applications, such as the analysis of crack-tip field, cylindrical and spherical cavity expansion in solids, and simulation and interpretation of experiments on the microscale, such as the twisting of thin copper wires and the micro-indentation tests on various metallic materials (see Fleck et al., 1994; Nix and Gao, 1998). Limited results are available in the literature in this regard, and are mostly application-specified and thus of restricted use. For example, Bleustein (1966) have derived formulations of the SGT in spherical coordinates in a study of the stress concentration at a spherical cavity. His results, however, are confined to the axi-symmetric case. Eshel and Rosenfeld (1970, 1975) have obtained formulations of the SGT for the cylindrical tube and cavity problems, but their discussions are limited to the plane strain case only. The formulations used by Chen et al. (1999) in an investigation of the asymptotic crack-tip field by strain gradient plasticity theory apply to plane strain case only. In a recent study of the torsional surface waves in a half space, following the approach of tensor analysis outlined in Malvern (1969), Georgiadis et al. (2000) have obtained formulations of the SGT with micro inertia in terms of cylindrical coordinates. The results, however, remain limited to the special case where only one component of displacements (u_θ therein) exists. While practical problems are often complex such that simplifications are not always achievable, it is highly desirable to have a set of general formulations of SGT in terms of curvilinear coordinates that are *general and complete* enough to cover most cases and may therefore lend great convenience of immediate use for future use. To the authors' knowledge, however, such formulations are still absent, and will thus be pursued in this note.

In view of the popularity of the Toupin–Mindlin strain gradient theory as discussed above, general formulations for this theory in orthogonal curvilinear coordinates will be derived, and will then be specified for two typical systems—cylindrical coordinates and spherical coordinates. It will be demonstrated that results in many existing studies can be covered as special cases by our formulations. In the subsequent derivation, the approach and the notation used by Eringen (1967) for the translation of conventional elasticity theories from rectangular coordinates to orthogonal curvilinear coordinates are closely followed. Wherever necessary, detailed explanations will be given on uncommonly used symbols and operations. To facilitate easy comprehension, the notation used in the paper is summarized as follows:

u_i	Displacement (components)
ε_{ij}	Strain tensor
η_{ijk}	Strain-gradient tensor
σ_{ij}	Cauchy stress tensor
τ_{ijk}	Higher-order stress tensor
λ,μ	Lamé constants
ξ_i	Elastic constants associated with gradient terms
l	Internal material length scale
T_k	Surface tractions

R_k	Higher-order surface tractions
$D_i = (\delta_{ik} - n_i n_k)\partial_k$	Surface gradient operator
$\bar{u}_i$	Displacements at the kinematic surface boundary
$\bar{e}_i$	Normal gradient of $\bar{u}_i$
n_i	Unit-normal vector
g_{kl}	Covariant components of the Euclidean metric tensor
g^{kl}	Contravariant components of the Euclidean metric tensor
g^{kk} or g_{kk}	The diagonal component of g^{ij} or g_{ij} (no sum on k)
$\det(\cdot)$	Determinant of a tensor
(,) at subscript	Partial differentiation (e.g.,$\sigma_{ij,k}$)
(;) at subscript	The covariant differentiation symbol (e.g.,$\sigma_{ij;k}$)
σ^i_j and τ^{ij}_k	Mixed form of Cauchy stress and Higher-order stress
ε^i_j and η^k_{ij}	Mixed components of strain and strain gradient
$\Sigma^i_k = \sigma^i_k - \tau^{ij}_{k;j}$	A generalised mixed-form second-order tensor
σ^*_{ij} and τ^*_{ijk}	Generalised stress and higher-order stress
$\left\{ \begin{matrix} k \\ lm \end{matrix} \right\}$	The Christoffel symbols of the second kind
$u^{(k)}, \varepsilon^{(i)}_{(j)}, \eta^{(k)}_{(i)(j)}$	The physical components of $u^k, \varepsilon^i_j, \eta^k_{ij}$
$u^{(k)}, \sigma^{(i)}_{(j)}, \tau^{(i)(j)}_{(k)}$	The physical components of $u^k, \sigma^i_j, \tau^{ij}_k$
δ_{ij} and δ^i_j	Covariant and mixed form of Kronecker delta
(r,θ,z)	Cylindrical coordinates
(r,θ,φ)	Spherical coordinates

2. Strain gradient theory in rectangular coordinates

The strain gradient theory to be treated here is based on Toupin's (1962) *Couple stress theory* and Mindlin's (1964) *elasticity theory with microstructure* by enforcing the relative deformation defined therein (the difference between the macro-displacement gradient and the micro deformation) to be zero. This theory can also be obtained by reducing the second-order strain gradient theory (or grade-three elasticity) proposed by Mindlin (1965) to the first order. In parallel with Mindlin's (1965) second-order strain gradient theory (or grade-three elasticity), the Toupin–Mindlin theory is sometimes also called *the first strain gradient theory* or *the linear theory of solids of grade two* (see, e.g., Toupin, 1964; Mindlin and Eshel, 1968; Eshel and Rosenfeld, 1970, 1975), where the term "grade" indicates the order of the space gradients operating on the displacement. In this theory, it is assumed that other than the conventional Eulerian strains ε_{ij} and Cauchy stresses σ_{ij}, strain gradients η_{ijk} and their work-conjugate higher-order stresses τ_{ijk} are also present in the material body, where the strains and strain gradients are respectively defined by:

$$\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2, \quad \eta_{ijk} = u_{k,ij} \tag{1}$$

where both ε_{ij} and η_{ijk} are symmetric with respect to the indices i and j. Accordingly, the Cauchy stress σ_{ij} and higher-order stress τ_{ijk} are also assumed to be symmetric about i and j. Consequently, under any small perturbations of strains and strain gradients, $\delta\varepsilon_{ij}$ and $\delta\eta_{ijk}$, the work deviation may be obtained by the two pairs of work-conjugates: $\delta W = \sigma_{ij}\,\delta\varepsilon_{ij} + \tau_{ijk}\delta\eta_{ijk}$. In addition, within the framework of linear elasticity, the following generalised Hooke's law between σ_{ij} and ε_{ij} and between τ_{ijk} and η_{ijk} are, respectively, assumed (c.f., e.g., Mindlin, 1964, 1965; Eshel and Rosenfeld, 1975):

$$\begin{cases} \sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \\ \tau_{ijk} = \xi_1 l^2(\eta_{ipp}\delta_{jk} + \eta_{jpp}\delta_{ik}) + \xi_2 l^2(\eta_{ppi}\delta_{jk} + 2\eta_{kpp}\delta_{ij} + \eta_{ppj}\delta_{ik}) \\ \qquad + \xi_3 l^2\eta_{ppk}\delta_{ij} + \xi_4 l^2\eta_{ijk} + \xi_5 l^2(\eta_{kji} + \eta_{kij}) \end{cases} \tag{2}$$

where λ and μ are conventional Lamé constants, while $\xi_i\,(i=1,5)$ are elastic constants associated with gradient terms in a material. l denotes an internal length scale resulted by the introduction of strain gradients, and is related to the dimension of microstructure in the material.

The governing equations and the associated boundary conditions for a gradient-dependent problem in rectangular coordinates can be obtained via variational principles (see Mindlin, 1964; Bleustein, 1966; Bleustein, 1967; Mindlin and Eshel, 1968; Germain, 1973). Consider a gradient-dependent material body with volume V and surface S. The following equilibrium equations are obtained for the gradient-dependent material body:

$$\sigma_{ik,i} - \tau_{ijk,ji} + f_k = 0 \tag{3}$$

where f_k denotes the body force, and here we neglect the higher-order body forces for simplicity. As for the corresponding boundary conditions, the external surface S may further be divided into two parts: the surface boundary S_σ for static forces, and the other is S_u for displacements. On the static force boundary S_σ, the following boundary conditions apply:

$$T_k = n_i(\sigma_{ik} - \partial_j\tau_{ijk}) - D_j(n_i\tau_{ijk}) + n_in_j(D_ln_l)\tau_{ijk} \tag{4}$$

$$R_k = n_in_j\tau_{ijk} \tag{5}$$

where, T_i and R_i are the surface tractions and higher-order surface tractions (or alternatively double traction), respectively. $D_i = (\delta_{ik} - n_in_k)\partial_k$, denoting the surface gradient operator. n_k is the normal vector in a local coordinate system. Eqs. (4) and (5), respectively, represent the conventional traction and higher-order traction conditions for a gradient-dependent material body.

In addition, on the kinematic surface S_u, let $\bar{u}_i$ denote the displacements. Note that only the normal gradients of $\bar{u}_i$ are independent of $\bar{u}_i$, while for known $\bar{u}_i$, its surface gradients are always known. Hence, totally six independent displacement boundary conditions are generally required for appropriately addressing a particular problem, e.g., the displacements $\bar{u}_i, i=1,2,3$ as well as their normal gradients along S_u should be initialized, which results in the following kinematic conditions for a gradient-dependent material body:

$$u_k = \bar{u}_k \text{ and } n_l\partial_l u_k = \bar{e}_k \text{ on } S_u \tag{6}$$

where $\bar{e}_k$ is the normal gradients of $\bar{u}_k$. Note that a rigorous derivation of the kinematic conditions in (6) has been given by Georgiadis and Grentzelou (2006) by using the principle of complementary virtual work and a Hellinger–Reissner-type variational principle.

3. Strain gradient theory in orthogonal curvilinear coordinates

In this section, general formulations of the aforementioned strain gradient elasticity in orthogonal curvilinear coordinates will be derived. The procedure closely follows that outlined in Eringen (1967) (pp. 204–210). A set of orthogonal curvilinear coordinates $x^k(k=1,3)$ are used to express the equilibrium equations and boundary conditions as presented in Eqs. (3)–(6). Let $g_{kl}(\mathbf{x})$[1] be the metric tensor in the curvilinear coordinates x^k. The square of the element of arc length ds is now given by

$$(\mathrm{d}s)^2 = g_{kl}\mathrm{d}x^k\mathrm{d}x^l \tag{7}$$

For the case of orthogonal curvilinear coordinates, we always have $g_{kl}=0$ when $k \neq l$, and thus

$$(\mathrm{d}s)^2 = g_{11}(\mathrm{d}x^1)^2 + g_{22}(\mathrm{d}x^2)^2 + g_{33}(\mathrm{d}x^3)^2 \tag{8}$$

And

$$g \equiv \det g_{kl} = g_{11}g_{22}g_{33}, \quad g^{\underline{kk}} = \frac{1}{g_{\underline{kk}}} \tag{9}$$

[1] g_{ij} are the covariant components of the Euclidean metric tensor which is defined by: $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ where $\mathbf{e}_i(i=1,3)$ denote right-handed base vectors. The contravariant components of the Euclidean metric tensor are denoted by g^{ij} and defined by $g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j$ where $\mathbf{e}^i(i=1,3)$ denote the reciprocal left-handed base vectors of $\mathbf{e}_i$. See p. 420 of Eringen (1967) for more detailed definitions.

where g is the determinant of g_{kl}. $g^{\underline{kk}}$ denotes an individual diagonal contravariant component of the Euclidean metric tensor, and $g_{\underline{kk}}$ a diagonal covariant component. The underscores are placed under the indices to temporarily suspend the Einstein summation. This convention is used throughout the paper.

Eringen (1967) suggested that the translation from rectangular coordinates to any curvilinear coordinates follows the following two rules: (a) The partial differentiation symbol (,) must be replaced by the covariant differentiation symbol (;), (b) The repeated indices must be on diagonal positions. Following these rules, the gradient-dependent equilibrium equations in curvilinear coordinates now have the following form instead of Eq. (3):

$$(\sigma^i{}_k - \tau^{ij}{}_{k;j})_{;i} - f_k = 0 \tag{10}$$

where $\sigma^i{}_k$ and $\tau^{ij}{}_k$ are, respectively, the mixed components of the stress tensor and higher-order stress tensor (see p. 462 of Eringen (1967) for definitions). For convenience of further manipulation, the following second-order tensor is introduced:

$$\Sigma^i{}_k = \sigma^i{}_k - \tau^{ij}{}_{k;j} \tag{11}$$

where an index following a semi-colon for a third-order tensor indicates the covariant partial differentiation as following, e.g. for $\tau^{ij}{}_k$:

$$\tau^{lm}{}_{n;q} = \tau^{lm}{}_{n,q} + \begin{Bmatrix} l \\ kq \end{Bmatrix}\tau^{km}{}_n + \begin{Bmatrix} m \\ kq \end{Bmatrix}\tau^{lk}{}_n - \begin{Bmatrix} k \\ qn \end{Bmatrix}\tau^{lm}{}_k \tag{12}$$

where $\begin{Bmatrix} k \\ lm \end{Bmatrix} \equiv \frac{\partial^2 z^n}{\partial x^l \partial x^m}\frac{\partial x^k}{\partial z^n}$ are the Christoffel symbols of the second kind where z^n denote rectangular coordinates. In orthogonal curvilinear coordinates, they have the following expression

$$\begin{Bmatrix} k \\ lm \end{Bmatrix} = \frac{1}{2g_{\underline{kk}}}\left(\frac{\partial g_{\underline{kk}}}{\partial x^m}\delta_{kl} + \frac{\partial g_{\underline{mm}}}{\partial x^l}\delta_{km} - \frac{\partial g_{\underline{ll}}}{\partial x_k}\delta_{lm}\right) \tag{13}$$

where $g_{\underline{kk}}$ has been explained in Eq. (9). δ_{kl} is the Kronecker delta. Note that the Christoffel symbols are not tensors, and are symmetric about l and m, such that $\begin{Bmatrix} k \\ lm \end{Bmatrix} = \begin{Bmatrix} k \\ ml \end{Bmatrix}$ (see Page 205 of Eringen (1967)). In analogy to Eq. (12), the covariant partial differentiation for a second-order tensor is:

$$t^k{}_{l;r} = t^k{}_{l,r} + \begin{Bmatrix} k \\ mr \end{Bmatrix} t^m{}_l - \begin{Bmatrix} m \\ rl \end{Bmatrix} t^k{}_m \tag{14}$$

Then we have

$$\Sigma^i{}_{k;i} = \Sigma^i{}_{k,i} + \begin{Bmatrix} i \\ ni \end{Bmatrix}\Sigma^n{}_k - \begin{Bmatrix} n \\ ik \end{Bmatrix}\Sigma^i{}_n \tag{15}$$

$$\sigma^i{}_{k;i} = \sigma^i{}_{k,i} + \begin{Bmatrix} i \\ ni \end{Bmatrix}\sigma^n{}_k - \begin{Bmatrix} n \\ ik \end{Bmatrix}\sigma^i{}_n \tag{16}$$

Thus the gradient-dependent equilibrium equations in curvilinear coordinates now present the following form:

$$\Sigma^i{}_{k,i} + \begin{Bmatrix} i \\ ni \end{Bmatrix}\Sigma^n{}_k - \begin{Bmatrix} n \\ ik \end{Bmatrix}\Sigma^i{}_n - f_k = 0 \tag{17}$$

By using Eqs. (12), (15) and (16), the equilibrium equations in Eq. (17) turn to be:

$$\begin{aligned}&\sigma^i{}_{k,i} - \tau^{ij}{}_{k,ji} - \begin{Bmatrix} i \\ mj \end{Bmatrix}\tau^{mj}{}_{k,i} - \begin{Bmatrix} j \\ mj \end{Bmatrix}\tau^{im}{}_{k,i} + \begin{Bmatrix} m \\ jk \end{Bmatrix}\tau^{ij}{}_{m,i} + \begin{Bmatrix} i \\ ni \end{Bmatrix}\left(\sigma^n{}_k - \tau^{nj}{}_{k,j} - \begin{Bmatrix} n \\ mj \end{Bmatrix}\tau^{mj}{}_k - \begin{Bmatrix} j \\ mj \end{Bmatrix}\tau^{nm}{}_k\right.\\ &\quad\left.+\begin{Bmatrix} m \\ jk \end{Bmatrix}\tau^{nj}{}_m\right) - \begin{Bmatrix} n \\ ik \end{Bmatrix}\left(\sigma^i{}_n - \tau^{ij}{}_{n,j} - \begin{Bmatrix} i \\ mj \end{Bmatrix}\tau^{mj}{}_n - \begin{Bmatrix} j \\ mj \end{Bmatrix}\tau^{im}{}_n + \begin{Bmatrix} m \\ jn \end{Bmatrix}\tau^{ij}{}_m\right) - f_k = 0\end{aligned} \tag{18}$$

In practical applications, the above equations are often conveniently expressed in terms of the physical components of the vectors and tensors involved. The physical components $\sigma^{(k)}{}_{(l)}$, $\tau^{(k)(l)}{}_{(m)}$ and $u^{(k)}$ of $\sigma^k{}_l$, $\tau^{kl}{}_m$ and u^k are, respectively, related to each other by the following relations:

$$\sigma^{k}{}_{l}=\sigma^{(k)}{}_{(l)}\sqrt{g_{\underline{ll}}/g_{\underline{kk}}},\quad \tau^{kl}{}_{m}=\tau^{(k)(l)}{}_{(m)}\sqrt{g_{\underline{mm}}/(g_{\underline{kk}}g_{\underline{ll}})},\quad u^{k}=u^{(k)}/\sqrt{g_{\underline{kk}}} \tag{19}$$

Upon substituting Eq. (19) into (18) and using (13), we can find the gradient-dependent equilibrium equations in orthogonal curvilinear coordinates in terms of the physical components. The final form of these equations using both cylindrical coordinates and spherical coordinates explicitly, if given, will be very convenient for direct use. They are hereby presented in the following two sections.

It is also useful to express the strains and strain gradients defined in Eq. (1) in terms of the physical components of the displacement vector. To this end, the strain tensor and strain gradient tensor are first expressed as:

$$\varepsilon^{i}{}_{j}=\frac{1}{2}(u^{i}{}_{;j}+g_{nj}g^{im}u^{n}{}_{;m}),\quad \eta^{k}{}_{ij}=u^{k}{}_{;ij} \tag{20}$$

where

$$u^{k}=g^{km}u_{m},\quad u^{k}{}_{;l}=u^{k}{}_{,l}+\begin{Bmatrix} k \\ ml \end{Bmatrix}u^{m} \tag{21}$$

$$u^{k}{}_{;lm}=u^{k}{}_{,lm}+\begin{Bmatrix} k \\ ql \end{Bmatrix}u^{q}{}_{,m}+\begin{Bmatrix} k \\ qm \end{Bmatrix}u^{q}{}_{,l}-\begin{Bmatrix} q \\ ml \end{Bmatrix}u^{k}{}_{,q}+\left(\begin{Bmatrix} k \\ qm \end{Bmatrix}\begin{Bmatrix} q \\ pl \end{Bmatrix}-\begin{Bmatrix} k \\ pq \end{Bmatrix}\begin{Bmatrix} q \\ ml \end{Bmatrix}\right)u^{p} \tag{22}$$

Using (19) and (20), the physical components of strain tensor and strain gradient tensor in orthogonal curvilinear coordinates can be expressed in terms of the physical components of **u** as follows:

$$\varepsilon^{(i)}{}_{(j)}=\varepsilon^{i}{}_{j}\frac{\sqrt{g_{\underline{ii}}}}{\sqrt{g_{\underline{jj}}}}=\frac{1}{2}\frac{\sqrt{g_{\underline{ii}}}}{\sqrt{g_{\underline{jj}}}}\left[\left(\left(\frac{u^{(i)}}{\sqrt{g_{\underline{ii}}}}\right)_{,j}+\begin{Bmatrix} i \\ mj \end{Bmatrix}\frac{u^{(m)}}{\sqrt{g_{\underline{mm}}}}\right)+g_{nj}g^{im}\left(\left(\frac{u^{(n)}}{\sqrt{g_{\underline{nn}}}}\right)_{,m}+\begin{Bmatrix} n \\ qm \end{Bmatrix}\frac{u^{q}}{\sqrt{g_{\underline{qq}}}}\right)\right] \tag{23}$$

$$\eta^{(k)}{}_{(i)(j)}=\eta^{k}{}_{ij}\frac{\sqrt{g_{\underline{kk}}}}{\sqrt{g_{\underline{ii}}g_{\underline{jj}}}}=\frac{1}{2}\frac{\sqrt{g_{\underline{kk}}}}{\sqrt{g_{\underline{ii}}g_{\underline{jj}}}}(u^{k}{}_{;ij}+u^{k}{}_{;ji}) \tag{24}$$

where

$$u^{k}{}_{;lm}=\left(\frac{u^{(k)}}{\sqrt{g_{\underline{kk}}}}\right)_{,lm}+\begin{Bmatrix} k \\ ql \end{Bmatrix}\left(\frac{u^{(q)}}{\sqrt{g_{\underline{qq}}}}\right)_{,m}+\begin{Bmatrix} k \\ qm \end{Bmatrix}\left(\frac{u^{(q)}}{\sqrt{g_{\underline{qq}}}}\right)_{,l}-\begin{Bmatrix} q \\ ml \end{Bmatrix}\left(\frac{u^{(k)}}{\sqrt{g_{\underline{kk}}}}\right)_{,q}+\left(\begin{Bmatrix} k \\ qm \end{Bmatrix}\begin{Bmatrix} q \\ pl \end{Bmatrix}-\begin{Bmatrix} k \\ pq \end{Bmatrix}\begin{Bmatrix} q \\ ml \end{Bmatrix}\right)\left(\frac{u^{(p)}}{\sqrt{g_{\underline{pp}}}}\right) \tag{25}$$

The corresponding boundary conditions presented in (4)–(6) take the following component forms in orthogonal curvilinear coordinates:

$$\begin{aligned}T^{(k)}=\;&n^{(i)}(\sigma^{(i)}{}_{(k)})^{*}-\frac{\sqrt{g_{\underline{jj}}}}{\sqrt{g_{\underline{kk}}}\sqrt{g_{\underline{pp}}}}(\delta^{(p)}{}_{(j)}-n^{(j)}n^{(p)})\\&\left[\left(\frac{\sqrt{g_{\underline{kk}}}}{\sqrt{g_{\underline{jj}}}}n^{(j)}\tau^{(i)(j)}{}_{(k)}\right)_{,p}+\begin{Bmatrix} j \\ mp \end{Bmatrix}\left(\frac{\sqrt{g_{\underline{kk}}}}{\sqrt{g_{\underline{mm}}}}n^{(j)}\tau^{(i)(m)}{}_{(k)}\right)-\begin{Bmatrix} m \\ pk \end{Bmatrix}\left(\frac{\sqrt{g_{\underline{mm}}}}{\sqrt{g_{\underline{jj}}}}n^{(i)}\tau^{(i)(j)}{}_{(m)}\right)\right]\\&+n^{(i)}n^{(j)}\left(\frac{\sqrt{g_{\underline{ll}}}}{\sqrt{g_{\underline{jj}}}}\left(\delta^{(p)}{}_{(l)}-n^{(l)}n^{(p)}\right)\left(\left(\frac{n^{(l)}}{\sqrt{g_{\underline{ll}}}}\right)_{,p}+\begin{Bmatrix} l \\ mp \end{Bmatrix}\frac{n^{(m)}}{\sqrt{g_{\underline{mm}}}}\right)\right)\tau^{(i)(j)}{}_{(k)}\end{aligned} \tag{26}$$

$$R^{(k)}=n^{(i)}n^{(j)}\tau^{(i)(j)}{}_{(k)} \tag{27}$$

$$u^{(k)}=\bar{u}^{(k)}\text{ and }\frac{\sqrt{g_{\underline{ll}}}}{\sqrt{g_{\underline{kk}}}}n^{(l)}\left[\left(\sqrt{g_{\underline{kk}}}u^{(k)}\right)_{,l}-\begin{Bmatrix} m \\ kl \end{Bmatrix}\left(\sqrt{g_{\underline{mm}}}u^{(m)}\right)\right]=\bar{e}^{(k)}\text{ on }S_{u} \tag{28}$$

where

$$\left(\sigma^{(i)}{}_{(k)}\right)^{*} = \sigma^{(i)}{}_{(k)} - \left[\tau^{(i)(j)}{}_{(k),(j)} + \begin{Bmatrix} i \\ mj \end{Bmatrix}\tau^{(m)(j)}{}_{(k)} + \begin{Bmatrix} j \\ mj \end{Bmatrix}\tau^{(i)(m)}{}_{(k)} - \begin{Bmatrix} m \\ jk \end{Bmatrix}\tau^{(i)(j)}{}_{(m)}\right].$$

It is also easy to verify, the constitutive relations presented in Eq. (2) remain the same as for orthogonal curvilinear coordinates:

$$\begin{cases} \sigma^{(i)}{}_{(j)} = \lambda\varepsilon^{(k)}{}_{(k)}\delta^{i}_{j} + 2\mu\varepsilon^{(i)}{}_{(j)} \\ \tau^{(i)(j)}{}_{(k)} = \xi_1 l^2\left(\eta^{(p)}{}_{(i)(p)}\delta^{j}{}_{k} + \eta^{(p)}{}_{(j)(p)}\delta^{i}{}_{k}\right) + \xi_2 l^2\left(\eta^{(i)}{}_{(p)(p)}\delta^{j}{}_{k} + 2\eta^{(p)}{}_{(k)(p)}\delta^{i}{}_{j} + \eta^{(i)}{}_{(p)(p)}\delta^{i}{}_{k}\right) \\ \qquad + \xi_3 l^2\eta^{(k)}{}_{(p)(p)}\delta^{i}{}_{j} + \xi_4 l^2\eta^{(k)}{}_{(i)(j)} + \xi_5 l^2\left(\eta^{(i)}{}_{(k)(j)} + \eta^{(j)}{}_{(k)(i)}\right) \end{cases} \tag{29}$$

It is noted that all the equilibrium equations, boundary conditions, strains and strain gradients will hereafter be expressed in component forms of vectors and tensors. And for the convenience of writing, we use the conventional component terms in place of the proceeding expressions for the physical components of all tensors and vectors, e.g., for cylindrical coordinates, using the common expression σ_{rr} in place of the component form $\sigma^{(r)}{}_{(r)}$, $\tau_{r\theta z}$ for $\tau^{(r)(\theta)}{}_{(z)}$, $\varepsilon_{\theta z}$ for $\varepsilon^{(\theta)}{}_{(z)}$, η_{rrr} for $\eta^{(r)}{}_{(r)(r)}$, and u_r for $u^{(r)}$, etc.

4. Strain gradient theory in cylindrical coordinates

The cylindrical coordinates (r, θ, z) as shown in Fig. 1 can be related to rectangular coordinates (x, y, z) by:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z \tag{30}$$

In cylindrical coordinates, the metric tensor g_{kl} has the following components:

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = 1, \quad g_{kl} = 0 \quad (k \neq l) \tag{31}$$

Consequently, the Christoffel symbols of the second kind in Eq. (13) have the following values in cylindrical coordinates:

$$\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 21 \end{Bmatrix} = \frac{1}{r}, \quad \begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -r, \quad \text{all others are zero} \tag{32}$$

In conjunction with Eqs. (18), (19), (31) and (32), the following gradient-dependent equilibrium equations in component form are obtained for cylindrical coordinates:

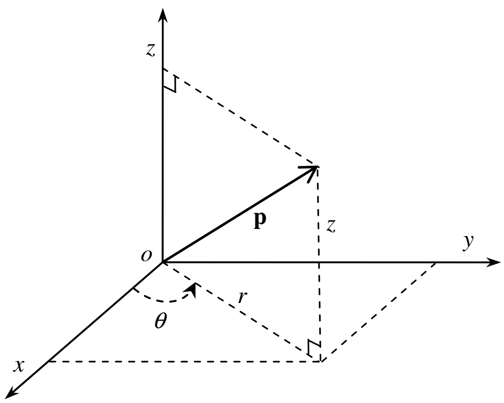

Fig. 1. Cylindrical coordinates in relation with rectangular coordinates.

$$\begin{cases} \frac{\partial \sigma^*_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma^*_{\theta r}}{\partial \theta} + \frac{\partial \sigma^*_{zr}}{\partial z} + \frac{1}{r}(\sigma^*_{rr} - \sigma^*_{\theta\theta}) = 0 \\ \frac{\partial \sigma^*_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma^*_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma^*_{z\theta}}{\partial z} + \frac{1}{r}(\sigma^*_{r\theta} + \sigma^*_{\theta r}) = 0 \\ \frac{\partial \sigma^*_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma^*_{\theta z}}{\partial \theta} + \frac{\partial \sigma^*_{zz}}{\partial z} + \frac{1}{r}\sigma^*_{rz} = 0 \end{cases} \tag{33}$$

where

$$\sigma^*_{rr} = \sigma_{rr} - \left(\frac{\partial \tau_{rrr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta r}}{\partial \theta} + \frac{\partial \tau_{rzr}}{\partial z} + \frac{1}{r}(\tau_{rrr} - \tau_{\theta\theta r} - \tau_{r\theta\theta})\right)$$
$$\sigma^*_{\theta\theta} = \sigma_{\theta\theta} - \left(\frac{\partial \tau_{\theta r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z\theta}}{\partial z} + \frac{1}{r}(\tau_{\theta r\theta} + \tau_{r\theta\theta} + \tau_{\theta\theta r})\right)$$
$$\sigma^*_{zz} = \sigma_{zz} - \left(\frac{\partial \tau_{zrz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{z\theta z}}{\partial \theta} + \frac{\partial \tau_{zzz}}{\partial z} + \frac{1}{r}\tau_{zrz}\right)$$
$$\sigma^*_{\theta r} = \sigma_{\theta r} - \left(\frac{\partial \tau_{\theta rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta r}}{\partial \theta} + \frac{\partial \tau_{\theta zr}}{\partial z} + \frac{1}{r}(\tau_{\theta rr} + \tau_{r\theta r} - \tau_{\theta\theta\theta})\right)$$
$$\sigma^*_{r\theta} = \sigma_{r\theta} - \left(\frac{\partial \tau_{rr\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta r\theta}}{\partial \theta} + \frac{\partial \tau_{zr\theta}}{\partial z} + \frac{1}{r}(\tau_{rr\theta} + \tau_{r\theta r} - \tau_{\theta\theta\theta})\right)$$
$$\sigma^*_{zr} = \sigma_{zr} - \left(\frac{\partial \tau_{zrr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{z\theta r}}{\partial \theta} + \frac{\partial \tau_{zzr}}{\partial z} + \frac{1}{r}(\tau_{zrr} - \tau_{z\theta\theta})\right)$$
$$\sigma^*_{rz} = \sigma_{rz} - \left(\frac{\partial \tau_{rrz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta rz}}{\partial \theta} + \frac{\partial \tau_{zrz}}{\partial z} + \frac{1}{r}(\tau_{rrz} - \tau_{\theta\theta z})\right)$$
$$\sigma^*_{\theta z} = \sigma_{\theta z} - \left(\frac{\partial \tau_{r\theta z}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta z}}{\partial \theta} + \frac{\partial \tau_{z\theta z}}{\partial z} + \frac{1}{r}(\tau_{r\theta z} + \tau_{\theta rz})\right)$$
$$\sigma^*_{z\theta} = \sigma_{z\theta} - \left(\frac{\partial \tau_{zr\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z\theta}}{\partial \theta} + \frac{\partial \tau_{zz\theta}}{\partial z} + \frac{1}{r}(\tau_{rz\theta} + \tau_{zr\theta})\right)$$

For the special case where all components of stresses and higher order stresses in the z direction vanish (e.g., Chen et al., 1999, for a plane strain crack tip field), which may be named as a generalised plane stress state in strain gradient theory, the above equilibrium equations are simplified to be the following two:

$$\begin{cases} \frac{\partial \sigma^*_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma^*_{\theta r}}{\partial \theta} + \frac{1}{r}(\sigma^*_{rr} - \sigma^*_{\theta\theta}) = 0 \\ \frac{\partial \sigma^*_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma^*_{\theta\theta}}{\partial \theta} + \frac{1}{r}(\sigma^*_{r\theta} + \sigma^*_{\theta r}) = 0 \end{cases} \tag{34}$$

Note that Equations in (34) are exactly the polar coordinates results obtained by Chen et al. (1999) for the plane crack tip field (Eqs. (37a) and (37b) therein). In the case of plain strain cylindrical cavity expansion, Eq. (34) can be further simplified as:

$$\frac{\partial \sigma^*_{rr}}{\partial r} + \frac{1}{r}(\sigma^*_{rr} - \sigma^*_{\theta\theta}) = 0 \tag{35}$$

which has been used in Zhao et al. (2007b) to solve the gradient-dependent stress and deformation field for the cavity expansion problem. It is easy to verify that in the case where the problem can be defined as an axi-symmetric plane strain problem such that there are only two nonzero displacements u_r and u_θ exist both of which depend only on r and θ, the formulations obtained in this section can be reduced to those as presented in Eshel and Rosenfeld (1975).

It is also useful to provide the expressions of physical components for strain tensor and strain gradient tensor in terms of the physical components of displacements in cylindrical coordinates. In view of Eqs. (23), (31) and (32), the following forms of physical components for strains in cylindrical coordinates are obtained:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{u_r}{r}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \quad \varepsilon_{r\theta} = \varepsilon_{\theta r} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right),$$
$$\varepsilon_{rz} = \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right), \quad \varepsilon_{\theta z} = \varepsilon_{z\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \tag{36}$$

which, are the same as for conventional mechanics without considering gradient effects. Furthermore, by using Eqs. (24)–(32), the following 27 physical components of strain gradients in cylindrical coordinates are obtained:

$$\eta_{rrr} = \frac{\partial^2 u_r}{\partial r^2}, \quad \eta_{\theta\theta r} = \frac{1}{r^2}\left(\frac{\partial^2 u_r}{\partial \theta^2} - 2\frac{\partial u_\theta}{\partial \theta} + r\frac{\partial u_r}{\partial r} - u_r\right),$$
$$\eta_{zzr} = \frac{\partial^2 u_r}{\partial z^2}, \quad \eta_{r\theta r} = \eta_{\theta rr} = \frac{1}{r}\frac{\partial^2 u_r}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{3u_\theta}{2r^2},$$
$$\eta_{rzr} = \eta_{zrr} = \frac{\partial^2 u_r}{\partial r \partial z}, \quad \eta_{\theta zr} = \eta_{z\theta r} = \frac{1}{r}\left(\frac{\partial^2 u_r}{\partial z \partial \theta} - \frac{\partial u_\theta}{\partial z}\right),$$
$$\eta_{rr\theta} = \frac{\partial^2 u_\theta}{\partial r^2} + \frac{u_\theta}{r^2}, \quad \eta_{\theta\theta\theta} = \frac{1}{r}\left(\frac{1}{r}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right),$$
$$\eta_{zz\theta} = \frac{\partial^2 u_\theta}{\partial z^2}, \quad \eta_{r\theta\theta} = \eta_{\theta r\theta} = \frac{1}{r}\left(\frac{\partial^2 u_\theta}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_r}{\partial r} - \frac{u_r}{2r}\right)$$
$$\eta_{rz\theta} = \eta_{zr\theta} = \frac{\partial^2 u_\theta}{\partial r \partial z}, \quad \eta_{\theta z\theta} = \eta_{z\theta\theta} = \frac{1}{r}\left(\frac{\partial^2 u_\theta}{\partial \theta \partial z} + \frac{\partial u_r}{\partial z}\right)$$
$$\eta_{rrz} = \frac{\partial^2 u_z}{\partial r^2}, \quad \eta_{\theta\theta z} = \frac{1}{r}\left(\frac{1}{r}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial u_z}{\partial r}\right),$$
$$\eta_{zzz} = \frac{\partial^2 u_z}{\partial z^2}, \quad \eta_{r\theta z} = \eta_{\theta rz} = \frac{1}{r}\left(\frac{\partial^2 u_z}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right),$$
$$\eta_{rzz} = \eta_{zrz} = \frac{\partial^2 u_z}{\partial r \partial z}, \quad \eta_{\theta zz} = \eta_{z\theta z} = \frac{1}{r}\frac{\partial^2 u_z}{\partial z \partial \theta}. \tag{37}$$

To obtain the corresponding boundary conditions in component form for cylindrical coordinates, we substitute Eqs. (31) and (32) into (26)–(28). The following are the obtained traction boundary conditions:

$$\begin{cases} T_r = n_p\sigma^*_{pr} - (\delta_{rq} - n_q n_r)n_p\tau_{pqr,r} - \frac{1}{r}(\delta_{\theta q} - n_q n_\theta)n_p\tau_{pqr,\theta} - (\delta_{zq} - n_q n_z)n_p\tau_{pqr,z} \\ \qquad + \frac{1}{r}n_p(n_r^2\tau_{prr} + (1 - n_\theta^2)(\tau_{p\theta\theta} - \tau_{prr}) + n_r n_z\tau_{pzr} - n_r n_\theta\tau_{pr\theta} - n_\theta n_z\tau_{pz\theta}) \\ T_\theta = n_p\sigma^*_{p\theta} - (\delta_{rq} - n_q n_r)n_p\tau_{pq\theta,r} - \frac{1}{r}(\delta_{\theta q} - n_q n_\theta)n_p\tau_{pq\theta,\theta} - (\delta_{zq} - n_q n_z)n_p\tau_{pq\theta,z} \\ \qquad + \frac{1}{r}n_p(n_r^2\tau_{pr\theta} - (1 - n_\theta^2)(\tau_{pr\theta} + \tau_{p\theta r}) + n_r n_\theta\tau_{prr} + n_z n_\theta\tau_{pzr} + n_r n_z\tau_{pz\theta}) \\ T_z = n_p\sigma^*_{pz} - (\delta_{rq} - n_q n_r)n_p\tau_{pqz,r} - \frac{1}{r}(\delta_{\theta q} - n_q n_\theta)n_p\tau_{pqz,\theta} - (\delta_{zq} - n_q n_z)n_p\tau_{pqz,z} \\ \qquad + \frac{1}{r}n_p((n_r^2 + n_\theta^2 - 1)\tau_{prz} + n_r n_z\tau_{pzz}) \end{cases} \tag{38}$$

$$R_r = n_p n_q\tau_{pqr}, \quad R_\theta = n_p n_q\tau_{pq\theta}, \quad R_z = n_p n_q\tau_{pqz} \tag{39}$$

where the repeated subscript p and q imply, respectively, an Einstein summation over r, θ and z. The displacement boundary conditions are obtained as follows:

$$\left.\begin{aligned} & u_r = \bar{u}_r, u_\theta = \bar{u}_\theta, u_z = \bar{u}_z \\ & n_r u_{r,r} + r n_\theta(u_{r,\theta} - u_\theta) + n_z u_{r,z} = \dot{e}_r \\ & n_r u_{\theta,r} + r n_\theta(u_{\theta,\theta} + u_r) + n_z u_{\theta,z} = \dot{e}_\theta \\ & n_r u_{z,r} + r n_\theta u_{z,\theta} + n_z u_{z,z} = \dot{e}_z \end{aligned}\right\} \text{ on } S_u \tag{40}$$

The constitutive relations may also be written with cylindrical coordinates directly according to Eq. (29):

$$\begin{cases} \sigma_{pq} = \lambda\varepsilon_{kk}\delta_{pq} + 2\mu\varepsilon_{pq} \\ \tau_{pqs} = \xi_1 l^2(\eta_{pkk}\delta_{qs} + \eta_{qkk}\delta_{ps}) + \xi_2 l^2(\eta_{kkp}\delta_{qs} + 2\eta_{skk}\delta_{pq} + \eta_{kkq}\delta_{ps}) \\ \qquad + \xi_3 l^2\eta_{kks}\delta_{pq} + \xi_4 l^2\eta_{pqs} + \xi_5 l^2(\eta_{sqp} + \eta_{spq}) \end{cases} \tag{41}$$

where p, q and s can be any of the three indices r, θ and z. The repeated index k denotes an Einstein summation over the three indices.

5. Strain gradient theory in spherical coordinates

Spherical coordinates(r,θ,φ) are related to rectangular coordinates (x,y,z) by (see Fig. 2):

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta \tag{42}$$

The corresponding metric tensor g_{kl} (g^{kl}) for spherical coordinates have the following components:

$$g_{11} = g^{11} = 1, \quad g_{22} = \frac{1}{g^{22}} = r^2, \quad g_{33} = \frac{1}{g^{33}} = r^2\sin^2\theta, \quad g_{kl} = 0 \ (k \neq l) \tag{43}$$

The Christoffel symbols of the second kind in Eq. (13) have the following values in spherical coordinates:

$$\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -r, \quad \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = -r\sin^2\theta, \quad \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 21 \end{Bmatrix} = \frac{1}{r}, \quad \begin{Bmatrix} 2 \\ 33 \end{Bmatrix} = -\sin\theta\cos\theta,$$
$$\begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 31 \end{Bmatrix} = \frac{1}{r}, \quad \begin{Bmatrix} 3 \\ 23 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 32 \end{Bmatrix} = \cot\theta, \quad \text{all others are zero.} \tag{44}$$

By using Eqs. (18), (19), (43), (44), the gradient-dependent equilibrium equations in spherical coordinates in terms of the physical components of the covariant tensors are obtained as follows:

$$\begin{cases} \frac{\partial\sigma^*_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma^*_{\theta r}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma^*_{\varphi r}}{\partial\varphi} + \frac{1}{r}\left(2\sigma^*_{rr} - \sigma^*_{\theta\theta} - \sigma^*_{\varphi\varphi} + \sigma^*_{\theta r}\cot\theta\right) - f_r = 0 \\ \frac{\partial\sigma^*_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma^*_{\theta\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma^*_{\varphi\theta}}{\partial\varphi} + \frac{1}{r}(2\sigma^*_{r\theta} + \sigma^*_{\theta r} + (\sigma^*_{\theta\theta} + \sigma^*_{\varphi\varphi})\cot\theta) - f_\theta = 0 \\ \frac{\partial\sigma^*_{r\varphi}}{\partial r} + \frac{1}{r}\frac{\partial\sigma^*_{\theta\varphi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma^*_{\varphi\varphi}}{\partial\varphi} + \frac{1}{r}(2\sigma^*_{r\varphi} + \sigma^*_{\varphi r} + 2\sigma^*_{\varphi\theta}\cot\theta) - f_\varphi = 0 \end{cases} \tag{45}$$

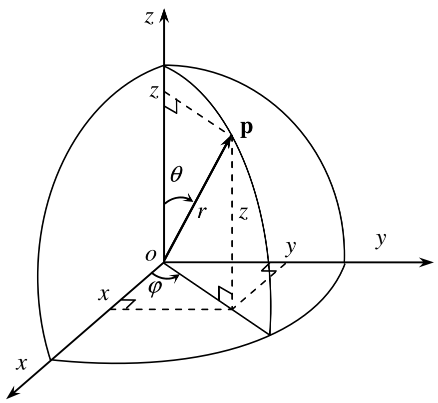

Fig. 2. Spherical coordinates in relation with rectangular coordinates.

where

$$\sigma^*_{rr} = \sigma_{rr} - \left(\frac{\partial\tau_{rrr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta r}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{r\varphi r}}{\partial\varphi} + \frac{1}{r}\left(2\tau_{rrr} - \tau_{\theta\theta r} - \tau_{r\theta\theta} - \tau_{\varphi\varphi r} - \tau_{r\varphi\varphi} + \tau_{r\theta r}\cot\theta\right)\right)$$
$$\sigma^*_{\theta r} = \sigma_{\theta r} - \left(\frac{\partial\tau_{\theta rr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\theta r}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\theta\varphi r}}{\partial\varphi} + \frac{1}{r}(2\tau_{\theta rr} + \tau_{r\theta r} - \tau_{\theta\theta\theta} - \tau_{\theta\varphi\varphi} + (\tau_{\theta\theta r} - \tau_{\varphi\varphi r})\cot\theta)\right)$$
$$\sigma^*_{\varphi r} = \sigma_{\varphi r} - \left(\frac{\partial\tau_{\varphi rr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\varphi\theta r}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\varphi\varphi r}}{\partial\varphi} + \frac{1}{r}(3\tau_{\varphi rr} - \tau_{\varphi\theta\theta} - \tau_{\varphi\varphi\varphi} + 2\tau_{\varphi\theta r}\cot\theta)\right)$$
$$\sigma^*_{\theta\theta} = \sigma_{\theta\theta} - \left(\frac{\partial\tau_{\theta r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\theta\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\theta\varphi\theta}}{\partial\varphi} + \frac{1}{r}(3\tau_{r\theta\theta} + \tau_{\theta\theta r} + (\tau_{\theta\theta\theta} - \tau_{\varphi\varphi\theta} - \tau_{\theta\varphi\varphi})\cot\theta)\right)$$
$$\sigma^*_{r\theta} = \sigma_{r\theta} - \left(\frac{\partial\tau_{rr\theta}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{r\varphi\theta}}{\partial\varphi} + \frac{1}{r}(2\tau_{rr\theta} + \tau_{r\theta r} - \tau_{\theta\theta\theta} - \tau_{\varphi\varphi\theta} + (\tau_{r\theta\theta} - \tau_{r\varphi\varphi})\cot\theta)\right)$$
$$\sigma^*_{\varphi\theta} = \sigma_{\varphi\theta} - \left(\frac{\partial\tau_{\varphi r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\varphi\theta\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\varphi\varphi\theta}}{\partial\varphi} + \frac{1}{r}(3\tau_{r\varphi\theta} + \tau_{\varphi\theta r} + (\tau_{\theta\varphi\theta} + \tau_{\varphi\theta\theta} - \tau_{\varphi\varphi\varphi})\cot\theta)\right)$$
$$\sigma^*_{\varphi\varphi} = \sigma_{\varphi\varphi} - \left(\frac{\partial\tau_{\varphi r\varphi}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\varphi\theta\varphi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\varphi\varphi\varphi}}{\partial\varphi} + \frac{1}{r}(3\tau_{r\varphi\varphi} + \tau_{\varphi\varphi r} + (2\tau_{\theta\varphi\varphi} + \tau_{\varphi\varphi\theta})\cot\theta)\right)$$
$$\sigma^*_{r\varphi} = \sigma_{r\varphi} - \left(\frac{\partial\tau_{rr\varphi}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta\varphi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{r\varphi\varphi}}{\partial\varphi} + \frac{1}{r}(2\tau_{rr\varphi} + \tau_{r\varphi r} - \tau_{\theta\theta\varphi} - \tau_{\varphi\varphi\varphi} + (\tau_{r\theta\varphi} + \tau_{r\varphi\theta})\cot\theta)\right)$$
$$\sigma^*_{\theta\varphi} = \sigma_{\theta\varphi} - \left(\frac{\partial\tau_{\theta r\varphi}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\theta\varphi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\theta\varphi\varphi}}{\partial\varphi} + \frac{1}{r}(3\tau_{r\theta\varphi} + \tau_{\theta\varphi r} + (\tau_{\theta\varphi\theta} + \tau_{\theta\theta\varphi} - \tau_{\varphi\varphi\varphi})\cot\theta)\right)$$

The physical components of strain tensor and strain gradient tensor in terms of the physical components of displacements in spherical coordinates will also be given here. The physical components for strains remain the same as in conventional theories:

$$\begin{aligned}\varepsilon_{rr} &= \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{u_r}{r}, \quad \varepsilon_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} + \frac{u_r}{r} + \frac{u_\theta}{r}\cot\theta, \quad \varepsilon_{r\theta} = \varepsilon_{\theta r}\\ &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right), \quad \varepsilon_{r\varphi} = \varepsilon_{\varphi r} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right), \quad \varepsilon_{\theta\varphi} = \varepsilon_{\varphi\theta}\\ &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_\varphi}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\varphi} - \frac{u_\varphi\cot\theta}{r}\right)\end{aligned} \tag{46}$$

The following 27 physical components for strain gradients in spherical coordinates are obtained by manipulating Eqs. (24)–(32):

$$\eta_{rrr} = \frac{\partial^2 u_r}{\partial r^2}, \quad \eta_{\theta\theta r} = \frac{1}{r^2}\left(\frac{\partial^2 u_r}{\partial\theta^2} + r\frac{\partial u_r}{\partial r} - 2\frac{\partial u_\theta}{\partial\theta} - u_r\right),$$
$$\eta_{\varphi\varphi r} = \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_r}{\partial\varphi^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} + \frac{1}{r^2}\frac{\partial u_r}{\partial\theta}\cot\theta - \frac{2}{r^2\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} - \frac{u_r}{r^2} - \frac{2u_\theta}{r^2}\cot\theta,$$
$$\eta_{\theta rr} = \eta_{r\theta r} = \frac{1}{r}\left(\frac{\partial^2 u_r}{\partial r\partial\theta} - \frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{\partial u_\theta}{\partial r} + \frac{3u_\theta}{2r}\right),$$
$$\eta_{\varphi rr} = \eta_{r\varphi r} = \frac{1}{r\sin\theta}\left(\frac{\partial^2 u_r}{\partial r\partial\varphi} - \frac{1}{r}\frac{\partial u_r}{\partial\varphi} - \frac{\partial u_\varphi}{\partial r}\sin\theta + \frac{3u_\varphi}{2r}\sin\theta\right),$$
$$\eta_{\varphi\theta r} = \eta_{\theta\varphi r} = \frac{1}{r^2\sin\theta}\left(\frac{\partial^2 u_r}{\partial\theta\partial\varphi} - \frac{\partial u_r}{\partial\varphi}\cot\theta - \frac{\partial u_\varphi}{\partial r}\sin\theta - \frac{\partial u_\theta}{\partial\varphi} + 2u_\varphi\cos\theta\right),$$
$$\eta_{rr\theta} = \frac{\partial^2 u_\theta}{\partial r^2} + \frac{u_\theta}{r^2}, \quad \eta_{\theta\theta\theta} = \frac{1}{r}\left(\frac{1}{r}\frac{\partial^2 u_\theta}{\partial\theta^2} + \frac{2}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right),$$
$$\eta_{\varphi\varphi\theta} = \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\theta}{\partial\varphi^2} - \frac{\cot\theta}{r^2\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} + \frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}\cot\theta + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}\cot^2\theta,$$

$$\begin{aligned}
&\eta_{\theta r\theta}=\eta_{r\theta\theta}=\frac{1}{r}\left(\frac{\partial^2 u_\theta}{\partial r\partial\theta}+\frac{\partial u_r}{\partial r}-\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}-\frac{u_r}{2r}\right),\\
&\eta_{\varphi r\theta}=\eta_{r\varphi\theta}=\frac{1}{r\sin\theta}\left(\frac{\partial^2 u_\theta}{\partial r\partial\varphi}-\frac{1}{r}\frac{\partial u_\theta}{\partial\varphi}-\left(\frac{\partial u_\varphi}{\partial r}-\frac{u_\varphi}{2r}\right)\cos\theta\right),\\
&\eta_{\varphi\theta\theta}=\eta_{\theta\varphi\theta}=\frac{1}{r^2\sin\theta}\left(\frac{\partial^2 u_\theta}{\partial\theta\partial\varphi}+\frac{\partial u_r}{\partial\varphi}-\frac{\partial u_\theta}{\partial\varphi}\cot\theta-\frac{\partial u_\varphi}{\partial\theta}\cos\theta+\frac{u_\varphi}{2}(3\cot\theta\cos\theta-\sin\theta)\right),\\
&\eta_{rr\varphi}=\frac{\partial^2 u_\varphi}{\partial r^2},\quad \eta_{\theta\theta\varphi}=\frac{1}{r}\left(\frac{1}{r}\frac{\partial^2 u_\varphi}{\partial\theta^2}+\frac{\partial u_\varphi}{\partial r}+\frac{1}{r\sin^2\theta}u_\varphi\right),\\
&\eta_{\varphi\varphi\varphi}=\frac{1}{r\sin\theta}\left(\frac{1}{r\sin\theta}\frac{\partial^2 u_\varphi}{\partial\varphi^2}+\frac{2}{r}\frac{\partial u_r}{\partial\varphi}+\frac{2}{r}\frac{\partial u_\theta}{\partial\varphi}\cot\theta+\frac{\partial u_\varphi}{\partial r}\sin\theta+\frac{\cos\theta}{r}\frac{\partial u_\varphi}{\partial\theta}-\frac{u_\varphi}{r\sin\theta}\right),\\
&\eta_{\theta r\varphi}=\eta_{r\theta\varphi}=\frac{1}{r}\left(\frac{\partial^2 u_\varphi}{\partial r\partial\theta}-\frac{1}{r}\frac{\partial u_\varphi}{\partial\theta}\right),\\
&\eta_{\varphi r\varphi}=\eta_{r\varphi\varphi}=\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial^2 u_\varphi}{\partial r\partial\varphi}-\frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}+\frac{\partial u_r}{\partial r}+\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)\cot\theta-\frac{u_r}{2r}\right),\\
&\eta_{\varphi\theta\varphi}=\eta_{\theta\varphi\varphi}=\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial^2 u_\varphi}{\partial\theta\partial\varphi}-\frac{\cot\theta}{\sin\theta}\frac{\partial u_r}{\partial\varphi}+\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial\theta}\cot\theta-\frac{u_\theta}{2\sin^2\theta}\right).
\end{aligned}\tag{47}$$

Again, substitution of (43) and (44) into (26)–(28) will result in the corresponding component-form boundary conditions for cylindrical coordinates, which are outlined as follows:

$$\begin{cases}
T_r=n_p\sigma^*_{pr}-(\delta_{rq}-n_qn_r)n_p\tau_{pqr,r}-\frac{1}{r}(\delta_{\theta q}-n_qn_\theta)n_p\tau_{pqr,\theta}-\frac{1}{r\sin\theta}(\delta_{\varphi q}-n_qn_\varphi)n_p\tau_{pqr,\varphi}\\
\quad+\frac{1}{r}(n_r^2+n_\theta^2+n_\varphi^2-3)n_p\tau_{prr}+\frac{1}{r}n_pn_q(n_r\tau_{pqr}-n_\theta\tau_{pq\theta}-n_\varphi\tau_{pq\varphi})\\
\quad+\frac{1}{r}n_p\tau_{pqq}+\frac{1}{r}n_rn_\theta n_p\cot\theta\tau_{prr}+\frac{1}{r}(n_\theta^2+n_\varphi^2-1)\cot\theta n_p\tau_{p\theta r}\\
T_\theta=n_p\sigma^*_{p\theta}-(\delta_{rq}-n_qn_r)n_p\tau_{pq\theta,r}-\frac{1}{r}(\delta_{\theta q}-n_qn_\theta)n_p\tau_{pq\theta,\theta}-\frac{1}{r\sin\theta}(\delta_{\varphi q}-n_qn_\varphi)n_p\tau_{pq\theta,\varphi}\\
\quad+\frac{1}{r}(n_r^2+n_\theta^2+n_\varphi^2-3)n_p\tau_{pr\theta}+\frac{1}{r}n_pn_q(n_\theta\tau_{pqr}+n_r\tau_{pq\theta}-n_\varphi\cot\theta\tau_{pq\varphi})\\
\quad+\frac{1}{r}n_p(\tau_{pr\theta}-\tau_{p\theta r}+\cot\theta\tau_{p\varphi\varphi})+\frac{1}{r}n_rn_\theta n_p\cot\theta\tau_{pr\theta}+\frac{1}{r}(n_\theta^2+n_\varphi^2-1)\cot\theta n_p\tau_{p\theta\theta}\\
T_\varphi=n_p\sigma^*_{p\varphi}-(\delta_{rq}-n_qn_r)n_p\tau_{pq\varphi,r}-\frac{1}{r}(\delta_{\theta q}-n_qn_\theta)n_p\tau_{pq\varphi,\theta}-\frac{1}{r\sin\theta}(\delta_{\varphi q}-n_qn_\varphi)n_p\tau_{pq\varphi,\varphi}\\
\quad+\frac{1}{r}(n_r^2+n_\theta^2+n_\varphi^2-3)n_p\tau_{pr\varphi}+\frac{1}{r}n_pn_q(n_\varphi\tau_{pqr}+n_r\tau_{pq\varphi}+n_\varphi\cot\theta\tau_{pq\theta})\\
\quad+\frac{1}{r}n_p(\tau_{pr\varphi}-\tau_{p\varphi r}-\cot\theta\tau_{p\varphi\theta})+\frac{1}{r}n_rn_\theta n_p\cot\theta\tau_{pr\varphi}+\frac{1}{r}(n_\theta^2+n_\varphi^2-1)\cot\theta n_p\tau_{p\theta\varphi}
\end{cases}\tag{48}$$

$$R_r=n_pn_q\tau_{pqr},\quad R_\theta=n_pn_q\tau_{pq\theta},\quad R_\varphi=n_pn_q\tau_{pq\varphi}\tag{49}$$

where the repeated subscript p and q imply, respectively, an Einstein summation over r, θ and φ. The displacement boundary conditions are obtained as follows:

$$\left.\begin{aligned}
&u_r=\bar{u}_r,u_\theta=\bar{u}_\theta,u_\varphi=\bar{u}_\varphi\\
&n_ru_{r,r}+rn_\theta(u_{r,\theta}-u_\theta)+rn_\varphi\sin\theta(u_{r,\varphi}-\sin\theta u_\varphi)=\dot{e}_r\\
&n_ru_{\theta,r}+rn_\theta(u_{\theta,\theta}+u_r)+rn_\varphi\sin\theta(u_{\theta,\varphi}-\cos\theta u_\varphi)=\dot{e}_\theta\\
&n_ru_{\varphi,r}+rn_\theta u_{\varphi,\theta}+rn_\varphi\sin\theta(u_{\varphi,\varphi}+\sin\theta u_r+\cos\theta u_\theta)=\dot{e}_\varphi
\end{aligned}\right\}\text{ on }S_u\tag{50}$$

The constitutive relations may be obtained for spherical coordinates by rewriting Eq. (29).

$$\begin{cases}
\sigma_{pq}=\lambda\varepsilon_{kk}\delta_{pq}+2\mu\varepsilon_{pq}\\
\tau_{pqs}=\xi_1l^2(\eta_{pkk}\delta_{qs}+\eta_{qkk}\delta_{ps})+\xi_2l^2(\eta_{kkp}\delta_{qs}+2\eta_{skk}\delta_{pq}+\eta_{kkq}\delta_{ps})\\
\qquad+\xi_3l^2\eta_{kks}\delta_{pq}+\xi_4l^2\eta_{pqs}+\xi_5l^2(\eta_{sqp}+\eta_{spq})
\end{cases}\tag{51}$$

where p, q and s can be any of the three indices r, θ and φ. The repeated index k denotes an Einstein summation over the three indices.

We note that for a radially symmetric (centro-symmetric) problem where there is only one displacement u_r which is a function of r only, the formulations derived in this section can be readily simplified to those obtained by Bleustein (1966).

6. Conclusions

The Toupin–Mindlin strain gradient theory has been reformulated and expressed in orthogonal curvilinear coordinates. Specific forms for cylindrical and spherical coordinates have been derived. Component form formulations for the corresponding equilibrium equations, boundary conditions, strains and strain gradients and constitutive relations have been given for the two typical curvilinear coordinate systems. These results have been shown to be general and complete and can be conveniently applied to a wide range of problems where orthogonal curvilinear coordinate descriptions are necessary in conjunction with the strain gradient theory, such as the analysis of crack-tip field in crystal materials, and the investigation of cylindrical cavity expansion and spherical void expansion in various solids, the interpretation of micro/nano indentation tests and bending or twisting tests of circular cylinder at small scales. Note that if further assumptions are made regarding the decomposition of strains and strain gradients and inelastic stress-strain relations (see, e.g., Fleck and Hutchinson, 1993, 1997, 2001; Chambon et al., 1996; Chambon et al., 1998, 2001, 2004; Zhao et al., 2005, 2006, 2007a,b; Zhao and Sheng, 2006), the current formulations can be extended to strain gradient plasticity without difficulty.

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