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Shakedown of Cohesive-frictional Non-homogeneous Soils under Moving Surface Loads

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ABSTRACT: In this paper, shakedown limits of non-homogeneous soils in a half-space under rolling and sliding surface loads are investigated. The inhomogeneity of the soil is described by a power law for the variation of Young's modulus with the depth. Mohr-Coulomb criterion is employed to characterise the cohesive-frictional behaviour of the soil. Analytical solutions to the elastic stress field under Hertzian surface contact are derived and then used in conjunction with Melan's shakedown theorem to compute the shakedown limits. The elastic graded effect of the soil on the various shakedown limits is investigated with reference to the corresponding homogeneous case. The results obtained can serve as benchmarks for future numerical shakedown analysis of pavements as well as valuable reference for practical design.

1 Introduction

Prediction of the long term behaviour of cohesive-frictional soils under cyclic moving surface loads is of significant importance to geotechnical and pavement engineering. It is difficult, however, to determine the pavement response to successive individual load applications by conducting step-by-step calculations as the processes are usually tedious and computationally expensive. Shakedown theory, on the other hand, can provide a rational and convenient way to determine the long term load-bearing capacity of the pavements. In particular, the elastic shakedown theorem proposed by Melan (1938) has been proved useful for design purposes in many structural and geotechnical applications. It has been repeatedly applied to the shakedown analysis of a cohesive-frictional half space under moving surface loads (see, e.g., Booker et al., 1985; Collins and Cliffe, 1987; Krabbenhoft et al, 2007; Zhao et al., 2007). In all these studies, the soil in the half space has been predominately assumed to be homogeneous for simplicity. Naturally deposited geomaterials, however, such as in-situ soils, clays and rocks are more often highly inhomogeneous, and their elastic parameters (e.g., Young's modulus) typically exhibit a graded effect for many soils. For example, Gibson (Gibson, 1967, Gibson et al., 1971, 1974) has employed a linear relation for the Young's modulus and the soil depth for characterising the so-called "Gibson soil". Many other practical soils, such as the Gault clay at Cambridge (UK), the London clay and chalk and Pliocence clays, the sensitive Ottawa clay and the Redcar glacial clays (UK), also demonstrate a highly inhomogeneous elastic properties (see, Giannakopoulos and Suresh, 1997). Therefore, it would be beneficial to account for the elastic inhomogeneity of the soil when performing shakedown analyses for engineering design. Numerical tools such as finite elements could be of use for the shakedown analysis of inhomogeneous soils, however, the success is mostly limited and the numerical results remain to be benchmarked. Analytical solutions to the shakedown limit that account for the elastic graded effect of soils are therefore highly desirable as they provide both benchmarks for the numerical modelling and reliable and convenient reference for practical designs.

2 Theoretical formulations

2.1 A power law model for the inhomogeneous Young's modulus in the soil

For the problem of rolling and sliding contact between vehicle wheels and a pavement, it is common to assume the deformation in the half space is plain strain such that all stresses are independent of the out-of-plane coordinates (say y for example, refer to Fig.1). The soil is assumed to be linearly elastic, inhomogeneous and locally isotropic within the context of small strains and small rotations. The Poisson's ratio is assumed to be constant over z for simplicity. While large values of Poisson's ratio ($\upsilon \rightarrow 0.5$) correspond to situations of undrained saturated soils, a typical value of 0.25 is adopted throughout this paper. In the literature, two models of elastic inhomogeneity of geomaterials are commonly used. The first one assumes the Young's modulus E of the material being dependent on the depth *z* according to a simple power law:

$$E = E_0 z^k \tag{1}$$

where E_0 has a dimension of [$m^{-1}Pa$]. For decaying values of stresses with depth, the power exponent, *k*, has to be in the interval: $0 \le k \le 1$. This model is believed to be suitable for dense sands and clay earth deposits. The second model of soil inhomogeneity assumes dependence of Young's modulus on the depth according to an exponential law $E = E_0 e^{\alpha z}$, where E_0 has the normal dimension of stress, and α has a dimension of [m^{-1}]. Variations of the Young's modulus over the depth according to these two models are depicted in Fig.1. In this paper, we will restrict our discussions to the power law case only.

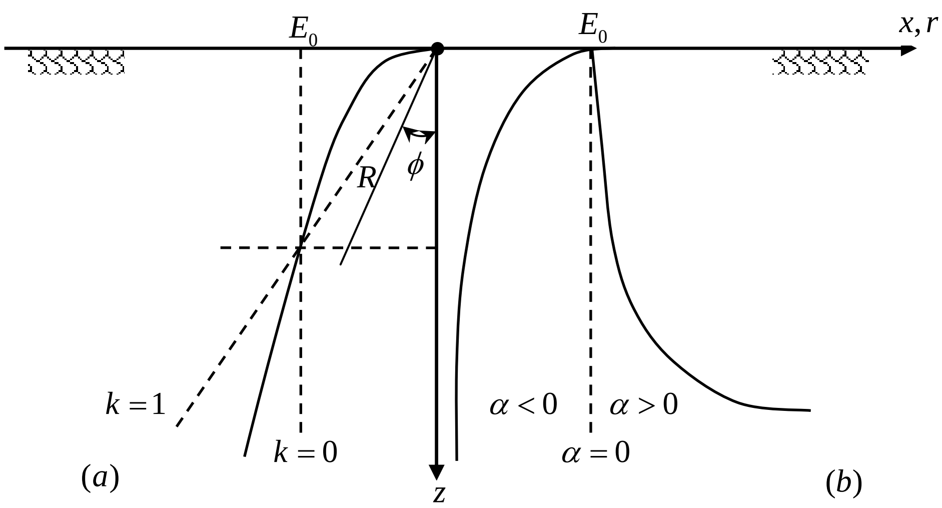

Figure 1. Two types of variation for Young's modulus in a half-space soil profile.
(a) the power law model, $E = E_0 z^k \left(0 \le k \le 1\right)$; (b) the exponential law model, $E = E_0 e^{\alpha z} \left(E_0 > 0\right)$

2.2 Contact approximation and elastic stress solution

Booker et al. (1985a, b) and Giannakopoulos and Pallot (2000) have investigated the elastic distribution in the half space under line and point loads by assuming a power law model for the Young's modulus as in Eq.(1). The stresses in a half space subjected to distributed normal and tangential tractions can be obtained by integrating the line load function over the traction surface (see, e.g., Johnson, 1985). By using the results obtained in Booker et al. (1985a) for line loads, the stresses in the elastic graded half space obeying the power law model and subjected to tractions as shown in Fig.2 can be obtained by the following integrals:

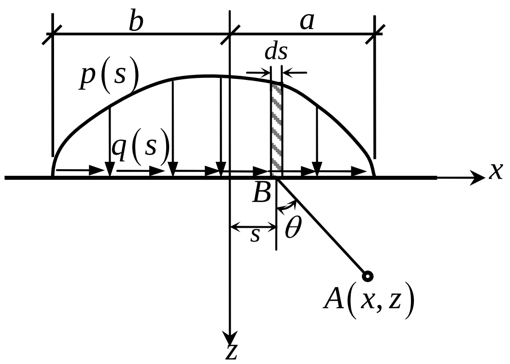

Figure 2. An elastic half space subjected to normal and tangential tractions over the strip ($-b \le x \le a$)

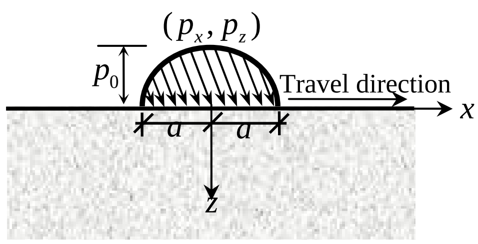

Figure 3. Approximation of the contact of pavement surface subject to rolling and sliding loading.

$$\begin{cases} \sigma_{xx} = -z^k F_{k\beta} \int_{-a}^{a} \dfrac{p(s)(x-s)^2 \cos\left[\beta \arctan\left((x-s)/z\right)\right]}{\left[(x-s)^2+z^2\right]^{(3+k)/2}} ds \\ \quad -\dfrac{(k+1)}{\beta} z^k F_{k\beta} \int_{-a}^{a} \dfrac{q(s)(x-s)^2 \sin\left[\beta \arctan\left((x-s)/z\right)\right]}{\left[(x-s)^2+z^2\right]^{(3+k)/2}} ds, \\ \sigma_{zz} = -z^{(k+2)} F_{k\beta} \int_{-b}^{a} \dfrac{p(s) \cos\left[\beta \arctan\left((x-s)/z\right)\right]}{\left[(x-s)^2+z^2\right]^{(3+k)/2}} ds \\ \quad -\dfrac{(k+1)}{\beta} z^{(k+2)} F_{k\beta} \int_{-b}^{a} \dfrac{q(s) \sin\left[\beta \arctan\left((x-s)/z\right)\right]}{\left[(x-s)^2+z^2\right]^{(3+k)/2}} ds, \\ \sigma_{xz} = -z^{(k+1)} F_{k\beta} \int_{-a}^{a} \dfrac{p(s)(x-s) \cos\left[\beta \arctan\left((x-s)/z\right)\right]}{\left[(x-s)^2+z^2\right]^{(3+k)/2}} ds \\ \quad -\dfrac{(k+1)}{\beta} z^{(k+1)} F_{k\beta} \int_{-a}^{a} \dfrac{q(s)(x-s) \sin\left[\beta \arctan\left((x-s)/z\right)\right]}{\left[(x-s)^2+z^2\right]^{(3+k)/2}} ds. \end{cases} \tag{2}$$

Where

$$F_{k\beta} = \frac{2^{(k+1)}(k+2)\Gamma\left(\dfrac{3+k+\beta}{2}\right)\Gamma\left(\dfrac{3+k-\beta}{2}\right)}{\pi\Gamma(3+k)} \tag{3}$$

$$\beta = \sqrt{(k+1)\left(1-\frac{k\nu}{1-\nu}\right)} \tag{4}$$

$\Gamma(\cdot)$ is the gamma function and υ is the Poisson's ratio. Note that when $\upsilon = 1/(k+2)$ such that $\beta = 1$, the Holl's case can be obtained (Holl, 1940), while the incompressible Gibson soil model is attained by letting $k = 1$, $\beta = 1$ and $F_{k\beta} = 2/\pi$.

For a homogeneous soil, the rolling and sliding contact between the vehicle wheels and road surface has been frequently approximated by a Hertzian contact (as shown in Fig.3), where the pressure distribution due to the moving roller is given by the following parabolic expression:

$$\begin{cases} p = p_0\sqrt{1-(x/a)^2} \\ q = \mu p \end{cases} \tag{5}$$

where p_0 denotes the maximum vertical pressure at $x = 0$. Coulomb's friction law has been assumed to relate the shear traction to the vertical pressure. We note that Hertzian contact approximation is considered to be generally adequate for small contact length, e.g., $a < 0.2R$, where R is the roller radius. In the case of heavy vehicles with low tyre pressure the contact length may be much larger, such that Hertzian contact approximation is no longer applicable, and trapezoidal form of pressure distribution is then widely used (see Collins and Cliffe, 1987; Zhao et al., 2007). Considering elastic graded soil we assume that the normal and tangential tractions for the contact length obey a similar parabolic distribution to one given by (5). However, in this case, these tractions are dependent on the elastic graded parameter k according to the following (c.f., Booker et al, 1985b; Giannakopoulos and Pallot, 2000):

$$\begin{cases} p = \dfrac{\Gamma(3+k)}{2^{(k+1)}\Gamma^2\left(\dfrac{3+k}{2}\right)} \dfrac{P}{2a}\sqrt{\left(1-(x/a)^2\right)^{(1+k)}} \\ q = \mu p \end{cases} \tag{6}$$

where P denotes the total vertical force applied to the pavement surface. To make the results comparable, we assume the total force P and contact length $2a$ are identical for both homogeneous and inhomogeneous cases. As we know, the maximum pressure p_0 in Eq.(5) for the homogeneous case is given by $p_0 = 2P / (\pi a)$. Hence Eq.(6) can be expressed in terms of p_0, viz:

$$\begin{cases} p = \zeta p_0 \sqrt{\left(1-(x/a)^2\right)^{(1+k)}} \\ q = \mu p \end{cases} \tag{7}$$

where

$$\zeta = \frac{\pi\Gamma(3+k)}{2^{(k+3)}\Gamma^2\left(\dfrac{3+k}{2}\right)} \tag{8}$$

With $k \to 0$ such that $\Gamma\left((3+k)/2\right) = \sqrt{\pi}/2$ and $\Gamma(3+k) = 2$, $\zeta = 1$, the homogeneous model is attained and the above expressions reduce to the Hertzian case in Eq.(5). Substituting the pressure distribution in Eq.(7) into Eq.(2) and implementing the integrations, we can obtain the elastic stresses in the elastic graded half space subjected to surface moving loads. Special attention is required when computing the stresses at the pavement surface, which will be discussed in the following section.

2.3 Mohr-Coulomb yield criterion for the pavement soil

The Mohr-Coulomb criterion is used to model the cohesive-frictional behaviour of the soil under plane strain conditions:

$$f(\sigma_{xx}, \sigma_{zz}, \sigma_{xz}) = \sqrt{(\sigma_{zz}-\sigma_{xx})^2 + 4\sigma_{xz}^2} - (\sigma_{zz}+\sigma_{xx})\sin\phi - 2c\cos\phi = 0 \tag{9}$$

where c is the cohesion and ϕ is the internal friction angle. In this paper, a convention of tension being positive is adopted.

2.4 Melan's static shakedown theorem

Melan's shakedown theorem states that *a sufficient condition for shakedown to occur is that a time-independent, self-equilibrated, residual stress field can be found such that, when added to the elastic stress field, it produces a combined stress field that nowhere and at no time violates the yield condition*. For the plane strain rolling and sliding contact problem, the permanent deformation and the residual stress distribution are independent of x. The equilibrium of the residual stresses thus implies that there is only one non-zero component ρ_{xx}, which is a function of z only. The yield condition on the *total* stresses for the plane strain half space then reads as follows:

$$f(\lambda\sigma_{ij}, \rho_{xx}) = \sqrt{(\lambda\sigma_{zz} - \lambda\sigma_{xx} - \rho_{xx})^2 + 4\lambda^2\sigma_{xz}^2} - (\lambda\sigma_{zz} + \lambda\sigma_{xx} + \rho_{xx})\sin\phi - 2c\cos\phi = 0 \tag{10}$$

where λ is the load factor. It is worth to mention here that in the case of two-point load domain with zero being one of them, a common mistake made in many past investigations on analytical shakedown analysis when using the Melan's shakedown theorem is that by merely setting $\partial f / \partial\rho_{xx} = 0$ and *enforcing inequality (10) only for elastic stresses corresponding to non-zero load point*. Then, the optimum residual stress is found to be:

$$\rho_{xx}^* = 2c\tan\phi + \lambda\left(\sigma_{xx} - \frac{1+\sin^2\phi}{\cos^2\phi}\sigma_{zz}\right) \tag{11}$$

and the corresponding shakedown limit is given by (see, also, Collins and Cliffe, 1987; Yu, 2005; Krabbenhøft *et al.*, 2007a):

$$\lambda_{sd} = \min \lambda \text{ s.t. } \left(\lambda > 0, \lambda = \frac{c}{|\sigma_{xz}| - \sigma_{zz} \tan \phi} \right) \tag{12}$$

It has been indicated in our recent papers (Zhao et al., 2007, Krabbenhøft *et al.*, 2007) that the shakedown limit so obtained is always an overestimated one when surface failure is critical, i.e., at large values of frictional coefficient. In this case, the optimum residual stress presented in Eq.(11) is found to be frequently well outside the yield surface correspondent to zero point load, which is not only theoretically unreasonable but also practically dangerous for design. It is therefore important to enforce yield constraints for all points of load domain when computing shakedown limit. It is also found to be the case that the equilibrium condition on the residual stress is sometimes neglected in numerical shakedown analysis of pavement, which could result in another overestimation for the static shakedown limit. Concluding all mentioned above, the Melan's static shakedown theorem for the pavement problem should take the following mathematical expression:

$$\lambda_{sd} = \min_{\lambda > 0} \lambda \text{ s.t. } \begin{cases} \partial \rho_{xx} / \partial x = 0, \\ \rho_{xx}^{-} \le \rho_{xx} \le \rho_{xx}^{+}, \\ f\left(\lambda \sigma_{ij}, \rho_{xx}\right) \le 0. \end{cases} \tag{12}$$

where

$$\rho_{xx}^{+} = -2c \tan\left(\frac{\phi}{2} - \frac{\pi}{4}\right), \quad \rho_{xx}^{-} = -2c \tan\left(\frac{\phi}{2} + \frac{\pi}{4}\right) \tag{13}$$

are the two bounds for ρ_{xx} obtained by enforcing $f\left(\rho_{xx}\right) = 0$. They actually represent the compressive and tensile strength limits of the soil, respectively.

3 Numerical results and discussions

3.1 Elastic stresses in the non-homogeneous soil

The integrands in Eq.(2) are too complex for the stresses to be derived analytically. Numerical solution in this case is the only option. To compute the stresses at subsurface points in the half space (z > 0), we employ a recursive adaptive Simpson quadrature integration method (see, Gander and Gautschi, 2000). An absolute tolerance of 1E-06 is used to control the integration error. The same procedure, however, cannot be applied for the computation of stresses at the pavement surface as the integration functions in Eq. (2) have singularities at z = 0 and serious numerical problems can arise. As will be demonstrated later, the accuracy of stresses at the surface is crucial for determining the reliable shakedown limit when surface failure becomes critical. They have to be computed very accurately. To resolve the situation a Fourier transformation is employed to express the surface stresses in terms of some hypergeometric functions first and then to evaluate the stresses subsequently. Details regarding this procedure are too lengthy to be reported here.

Fig.4 shows the stresses obtained for a roller coefficient μ =1.0 for both the homogeneous and non-homogeneous cases. In the non-homogeneous case, the elastic graded index k is set to a value of 0.3. The Poisson's ratio adopts a value of 0.25 as said before. As can be seen, the relative magnitudes of stresses in both cases are comparable. The specific distributions, however, are slightly different. For σ_{xx}, the non-homogeneous case yields a distribution with a certain degree of stress concentration at the surface. Further computations show that with the soil becoming even linear (k tends to 1), this concentration becomes more pronounced. In contrast, the inhomogeneity of Young's modulus seems to have little impact on the other stresses, σ_{zz} and σ_{xz}, in comparison with the corresponding homogeneous case. The surface stresses, in particular σ_{xx}, are crucial in predicting the failures of frictional contact. As such we would like to have a further look on the influence of inhomogeneity on the stresses at the surface. In Fig.5 we present the variation of surface stresses at different value of k. We adopt μ = 0.5 for this case. Note that the "Hertzian stress" denoted in the figure is that obtained for the homogeneous case which is in the literal meaning the solution derived by Hertz. As is shown in Fig.5, the elastic inhomogeneity has a significant impact on the surface stress distribution within the contact length. In particular, greater compressive stresses are developed at the contact surface for inhomogeneous case comparing to homogeneous one. And the larger the value of k, the greater this difference is, as illustrated in Fig.4. Outside the contact region, both σ_{zz} and σ_{xz} vanish just analogous to the homogeneous case, whereas σ_{xx} tends to zero at infinity.

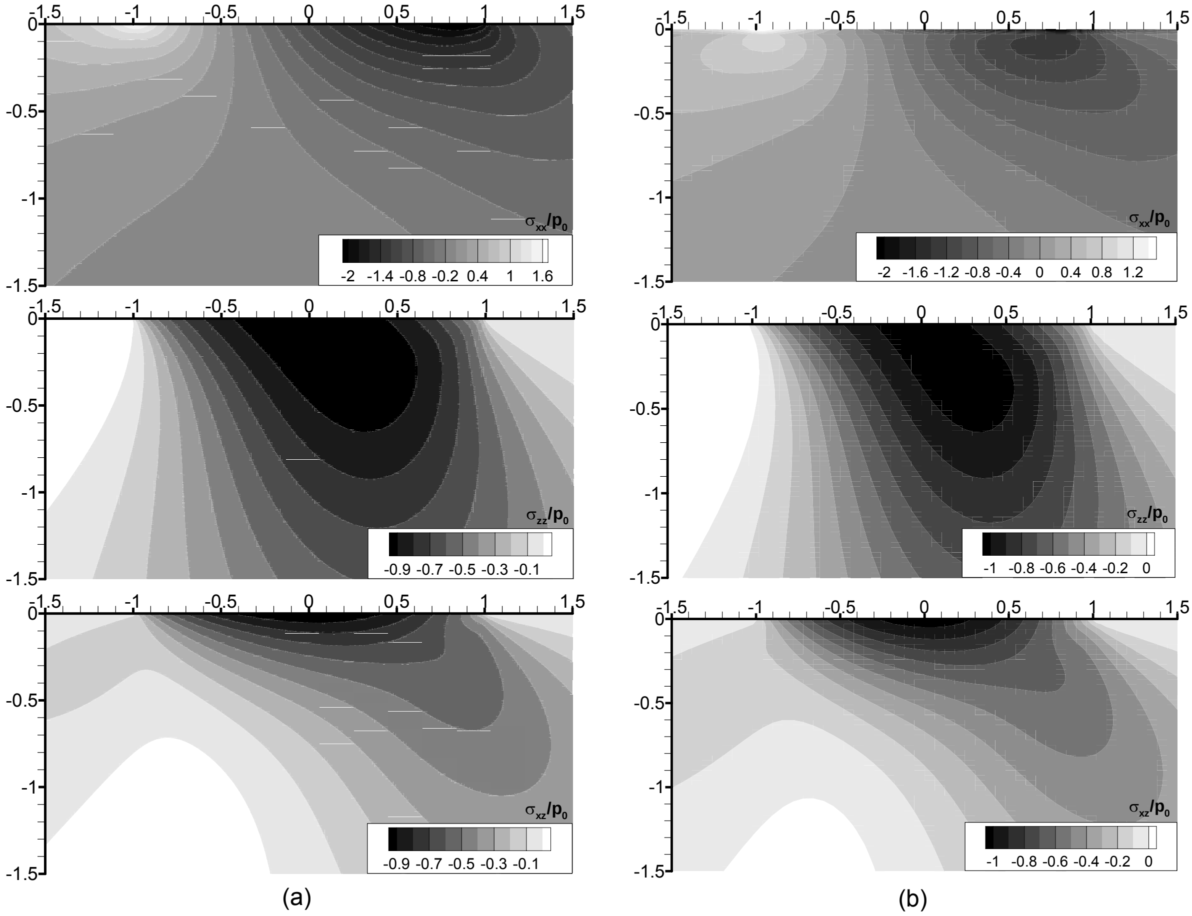

Figure 4. Elastic stresses in the half space under rolling and sliding contact. (a) Homogeneous case; (b) Non-homogeneous case with $k = 0.4$ and $\upsilon = 0.25$.

3.2 Shakedown limits

Using the elastic stress field obtained above, we compute next the shakedown limit for the non-homogeneous half space under rolling and sliding surface contact by enforcing Eq.(12) with the yield function given by Eq.(9). The procedure follows closely those presented in Zhao et al.(2007). Fig.6 presents the shakedown limits for various values of k at a frictional angle of 15° against the corresponding ones in the homogeneous case. Elastic limits are also included in the figure for comparison. As is shown, in all cases, the shakedown curve exhibits an obvious transition with the increase of roller coefficient μ, which distinguishes the failures of subsurface and surface nature. Further computations show the critical value of μ for this transitional point is ranged from 0.07 to 0.38 for various frictional angles (see also Zhao et al., 2007). The impact of inhomogeneity of the soil on the shakedown limit is evident from Fig.6. With the increase of k, the shakedown limit at small values of μ is increased compared to the homogeneous case (k = 0). This increase is significant especially for cases where subsurface failure is critical. When μ is large and surface failure prevails, shakedown limits for the non-homogeneous case remain larger than those for the homogeneous case whereas the difference is not so significant. It is also observed that the elastic inhomogeneity affects the yielding in the soil significantly. The elastic limit reduces at small values of μ with k being increased. A reverse trend is found for large values of μ.

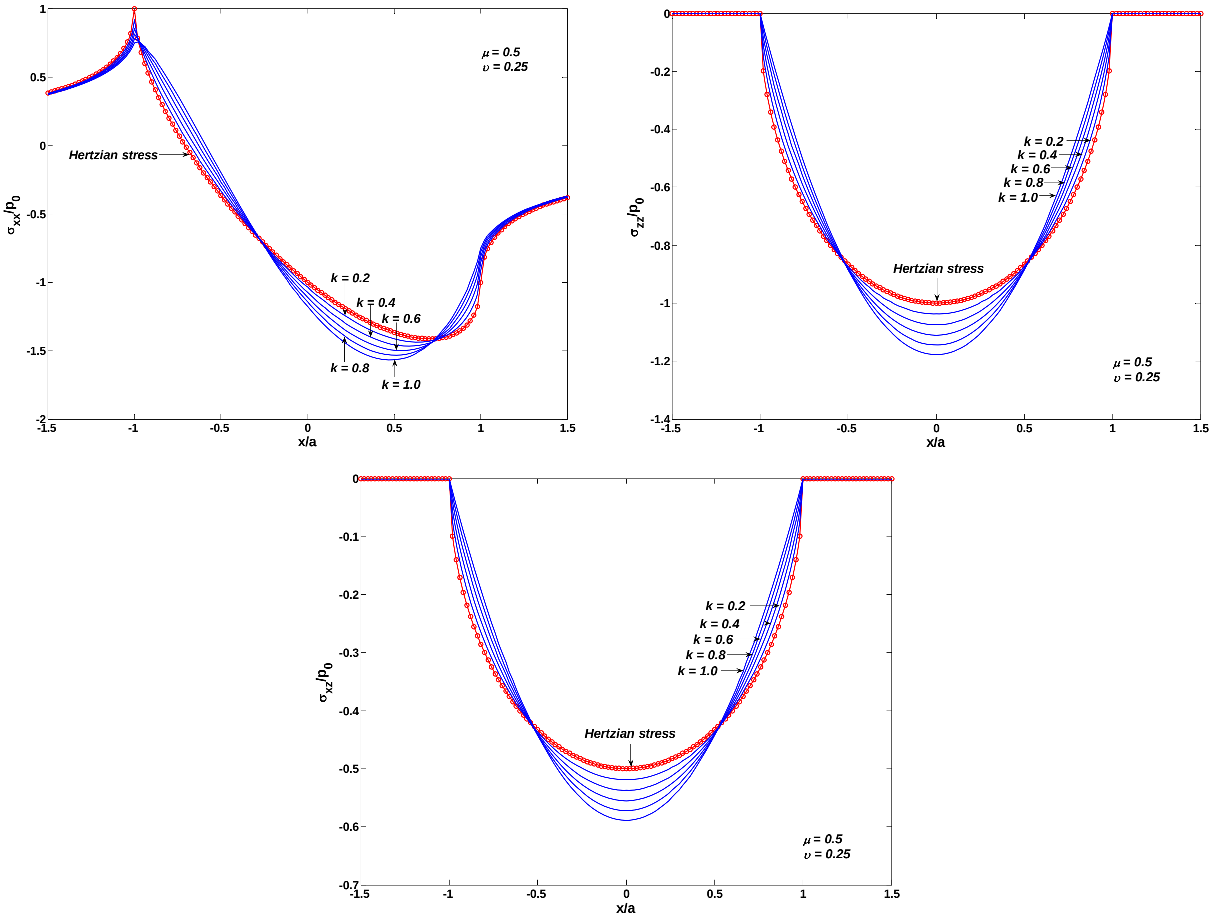

Figure 5. Stresses at the surface of the half space under rolling and sliding contact (μ = *0.5* and υ = 0.25).

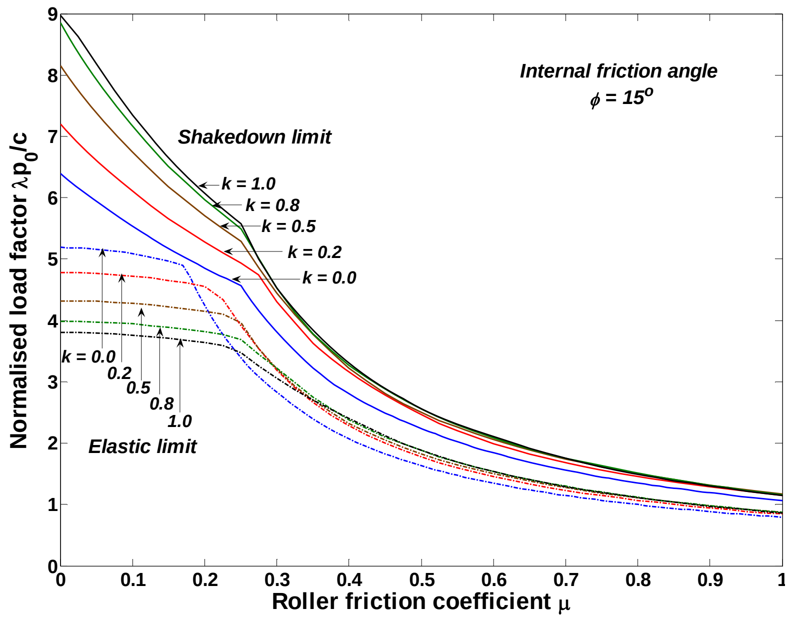

Figure 6. Shakedown and elastic limits for the non-homogeneous half space ($\phi = 15^{\circ}$).

4 Conclusions

In this paper, the stress distribution and shakedown of a non-homogeneous half space under surface rolling and sliding contact are thoroughly investigated. Mohr-Coulomb criterion is employed for describing the cohesive-frictional behaviour of yielding in the soil. It is found that the power model of elastic inhomogeneity leads to a rather different horizontal stress distribution comparing to the homogeneous case. As a conclusion, the considered form of soil inhomogeneity results in a greater shakedown limit than observed for homogeneous soil. The increase is most significant for the case when the roller frictional coefficient is small and thus subsurface failures dominate.

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